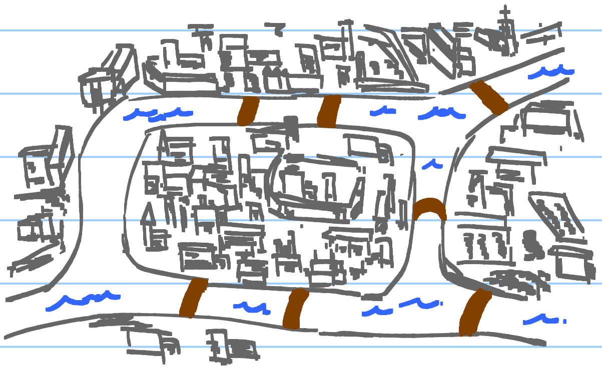
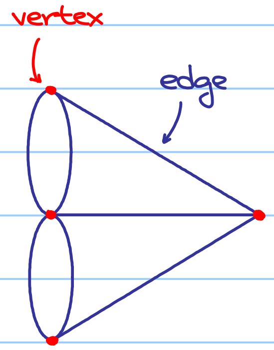
§ 10 Graphs

Origin of Graph theory

Königsberg Bridge Problem



formulate



vertex : land

edge : bridges

Can one cross each of the seven bridges exactly once and return to the starting point ?
Euler is the first mathematican to formulate and answer the question in terms of graphs. (Travel all edges exactly once and return to the initial vertex)

Definitions

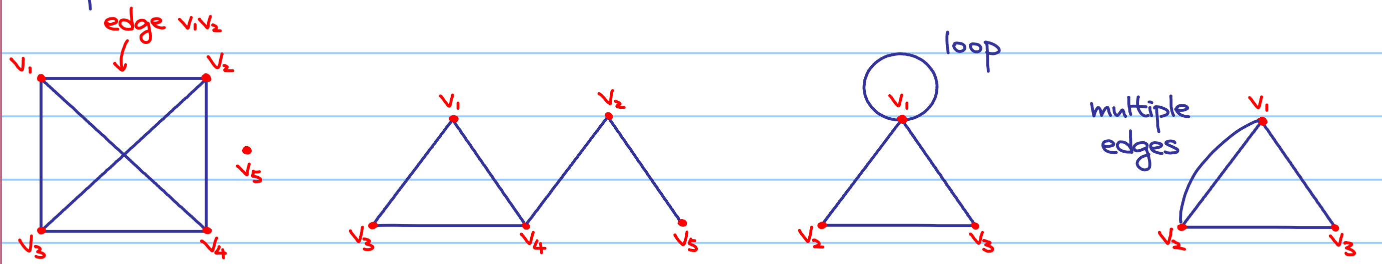
Definition 10.1

A simple graph $G=(V,E)$ consists of a nonempty set of vertices V and a set of edges E. Each edge has two distinct vertices associated with it, called its endpoints. There is at most one edge joining a pair of distinct vertices.

(Remark: Define $\Delta = \{(a,a) : a \in V\} \subseteq V \times V$ and define $\sim$ on $(V\times V)/\Delta$ such that $(a,b)\sim(b,a)$.
Then $E \subseteq ((V\times V)\setminus\Delta)/\sim$.)

Instead of writing down the elements of V and E, we usually represent G as below.

Example 10.1



(a) (b) (c) (d)

(a), (b) are simple graphs while (c), (d) are not

If loops and multiple edges are allowed, it is called a general graph.

(Remark: If general graphs are considered, E may be a multiset)

Definition 10.2

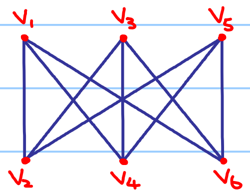
If V and E are finite, then $G=(V,E)$ is called a finite graph, otherwise it is called an infinite graph.

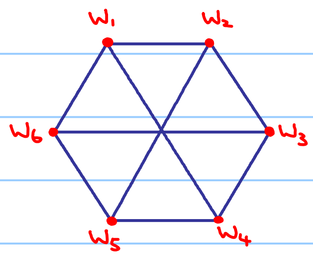
In this notes, a graph means a finite general graph.

Definition 10.3

If $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are graphs, G_1 and G_2 are said to be isomorphic if there exists a bijective functions $f_V: V_1 \to V_2$ and $f_E: E_1 \to E_2$ such that $e \in E_1$ is joining $v, w \in V_1$ if and only if $f_E(e) \in E_2$ is joining $f_V(v), f_V(w) \in V_2$.

Example 10.2





By considering $v_i \longleftrightarrow w_i$, the above graphs are isomorphic.

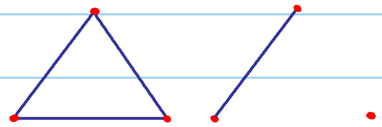
Definition 10.4

If $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are two graphs, the union graph $G_1 \cup G_2$ is defined as $G=G_1 \cup G_2 := (V_1 \cup V_2, E_1 \cup E_2)$.

A graph is said to be connected if it cannot be expressed as the union of two graphs, otherwise it is said to be disconnected.

Any disconnected graph can be expressed as the union of connected graphs, each of which is called a component of G.

Example 10.3



The above graph consists of 3 components.

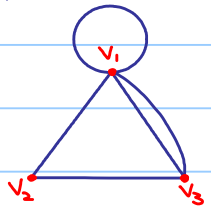
(Think: Why is the graph consists of only one vertex with no edge connected ?)

Definition 10.5

Two vertices are said to be adjacent if there is an edge joining them.

The degree of a vertex v, denoted by $\deg(v)$, is the number of edges incident with v.

Example 10.4



$\deg(v_1) = 5$

$\deg(v_2) = 2$

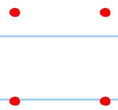
$\deg(v_3) = 3$

Lemma 10.1 (Handshaking Lemma)

$\sum_{v \in V} \deg(v) = 2|E|$. $\quad$ Σ # handshakings of each person = 2 × # handshakings

Examples

Null graphs N_n : A graph with n vertices but no edge.

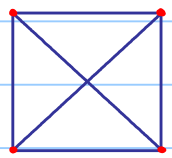


N_4

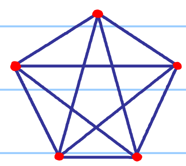
Complete graphs K_n : A graph with n vertices and each pair of distinct vertices are joined by exactly one edge.

Therefore, K_n has $C^n_2 = \frac{n(n+1)}{2}$ edges.

examples:



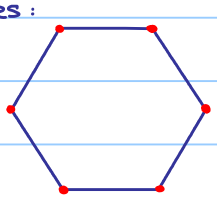
K_4

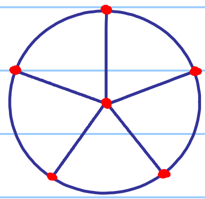


K_5

Cycle graphs, path graphs and wheels:

examples:





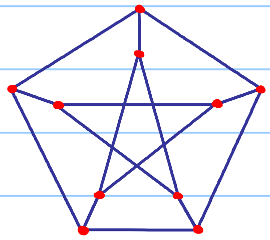
C_6 Cycle graph | P_6 Path graph | W_6 Wheel

Regular graphs: Degree of all vertices are the same

examples:

Petersen graph

regular of degree 3

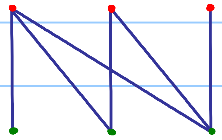


Cycle graph C_n

regular of degree 2

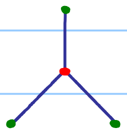
Bipartite graphs: Set of vertices V can be split into two disjoint sets A and B, each edge joins a vertex in A and a vertex in B.

example:

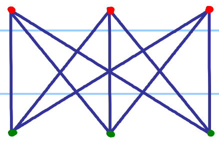


Complete bipartite graphs $K_{r,s}$: A bipartite graph with $|A| = r$, $|B| = s$ and each vertex in A is joined to every vertex in B by exactly one edge. Therefore $K_{r,s}$ has rs edges.

examples:



$K_{1,3}$



$K_{3,3}$

Paths and Cycles

Defintion 10.6

Let G be a graph.

A walk in G is a finite sequence of edges of the form $v_0v_1, v_1v_2, \cdots, v_{m-1}v_m$, also denoted by $v_0 \to v_1 \to v_2 \to \cdots \to v_m$, where v_i and v_{i+1} are adjacent or identical.

(If there exist more than one edge joining v_i and v_{i+1}, then which edge is travelled has to be specified)

The length is m, v_0 and v_m are initial and final vertex respectively

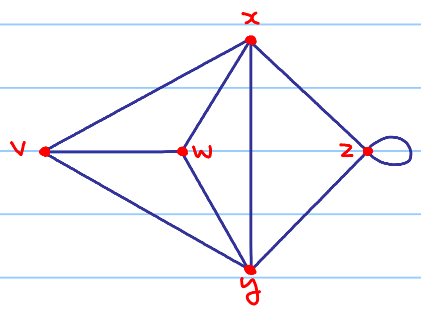
A trail is a walk in which all the edges are distinct.

A path is a trail in which all the vertices, but possibly $v_0 = v_m$, are distinct.

A path or trail is closed if $v_0 = v_m$.

A cycle is a closed path containing at least one edge.

Example 10.5



$v \to w \to x \to y \to z \to z \to x$ is a trail

$v \to w \to x \to y \to z$ is a path

$v \to w \to x \to y \to z \to x \to v$ is a closed trail

$v \to w \to x \to y \to v$ is a cycle

Proposition 10.1 (Connected = Path Connected)

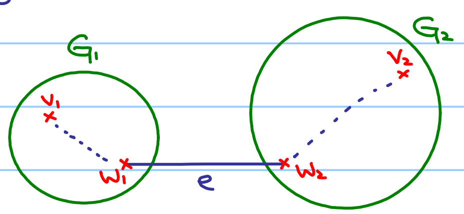
A graph is connected if and only if there is a path between each pair of vertices.

proof:

Let $G = (V, E)$ be a graph.

"$\Leftarrow$" Suppose the contrary. Let $G = G_1 \cup G_2$, $v_1 \in V_1$ and $v_2 \in V_2$.

By assumption, there exists a path $v_1 \to \cdots \to v_2$, but there exists an edge e joining $w_1 \in V_1$ and $w_2 \in V_2$, so $e \notin E_1 \cup E_2$ (Contradiction)



"$\Rightarrow$" Suppose the contrary. There exist $v_1, v_2 \in V$ such that no path joins v_1 and v_2.

Let $V_1 = \{v : \exists \text{ path joining } v \text{ and } v_1\}$ and let $V_2 = V \setminus V_1$. By assumption, $v_2 \in V_2$.

Let $E_1 = \{e \in E : \text{both endpoints} \in V_1\}$ and let $E_2 = E \setminus E_1$.

Note: If $e \in E_2$, both endpoints are in V_2. Otherwise, if e joins $w_1 \in V_1$ and $w_2 \in V_2$

$v_1 \to \cdots \to w_1 \xrightarrow{e} w_2$ is a path joining v_1 and w_2, so $w_2 \in V_1$ which is a contradiction.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then $G = G_1 \cup G_2$ which implies G is disconnected

(Contradiction).

Proposition 10.2

Let G be a simple graph with n vertices. If G has k components, then

$$n-k \le |E| \le \frac{(n-k)(n-k+1)}{2}.$$

proof:

Lower bound:

If we add one edge to a graph, the number of components is either unchanged or decreased by 1 (Why?)

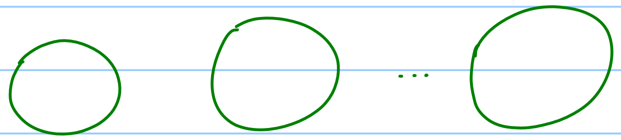
We start from the null graph N_n, there are n components.

The minimum number of edges to add to obtain a graph with k components is $n-k$.

Upper bound:

To obtain the upper bound of $|E|$,

we may assume each component is a complete graph



K_{r_1} K_{r_2} $\cdots$ K_{r_k}

$r_1 + r_2 + \cdots + r_k = n$.

Claim: The upper bound of $|E|$ is attained in the extreme case that all components except the last are N_1 and the last component is K_{n-k+1}

$$\underbrace{\bullet \quad \bullet \quad \cdots \quad \bullet}_{k-1} \qquad K_{n-k+1}$$

$$\sum_{i=1}^{k} \# \text{ edges of } K_{r_i}$$
$$= \sum_{i=1}^{k} \frac{r_i(r_i-1)}{2}$$
$$= \frac{1}{2}\sum_{i=1}^{k}[(r_i-1)^2 + r_i - 1]$$
$$= \frac{1}{2}\left[\left(\sum_{i=1}^{k}(r_i-1)^2\right) + n-k\right]$$
$$\leq \frac{1}{2}\left[\left(\sum_{i=1}^{k}(r_i-1)\right)^2 + n-k\right]$$
$$= \frac{1}{2}[(n-k)^2+(n-k)]$$
$$= \frac{(n-k)(n-k+1)}{2}$$
$$= \text{number of edges of } K_{n-k+1}$$

Corollary 10.1

A simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges must be connected.

Definition 10.7

A connected graph G is Eulerian if there exists a closed trail containing every edge of G and such trail is called an Eulerian trail.

A connected, non-Eulerian graph G is semi-Eulerian if there exists a trail containing every edge of G.

Question: Necessary and sufficient conditions for a graph to be Eulerian?

Before that, we need:

Lemma 10.2

If G is a graph such that $\deg(v) \geq 2$ for all $v \in V$, then G contains a cycle.

proof:

Trivial, if G has loops or multiple edges.

Assume G is a simple graph.

Take $v \in V$, $\deg(v) \geq 2 \Rightarrow$ there exists $v_1 \in V$ such that v_1 is adjacent to v

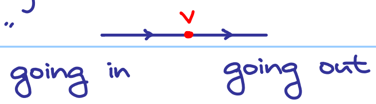
$\deg(v_1) \geq 2 \Rightarrow$ there exists $v_2 \in V$ such that v_2 is adjacent to v_1. $\cdots$

Repeating this and construct a walk $v \to v_1 \to v_2 \to \cdots$, since there are only finitely many vertices, eventually we have $v_i = v_j$ for some $i<j$.

Theorem 10.1 (Euler 1736)

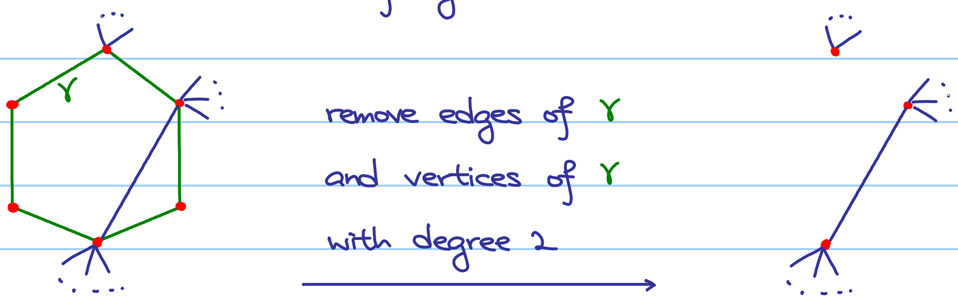
A connected graph G is Eulerian if and only if the $\deg(v)$ is even for all $v \in V$.

proof:

"$\Rightarrow$" 

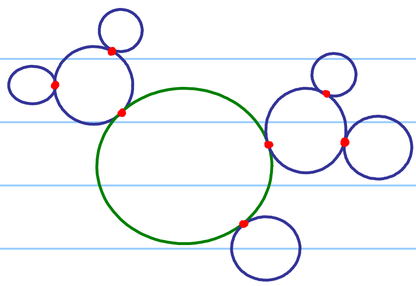
make a contribution of 2 to $\deg(v)$

"$\Leftarrow$" Lemma 10.2 $\Rightarrow$ Existence of cycle γ



Every remaining vertex is having even degree.

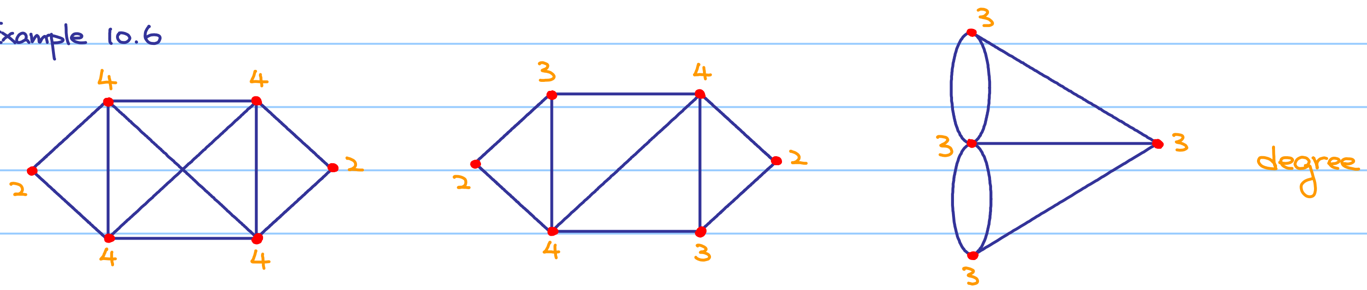
Repeating the above to obtain cycles until no edge remains, recombining the cycles to form an Eulerian trail.



Corollary 10.2

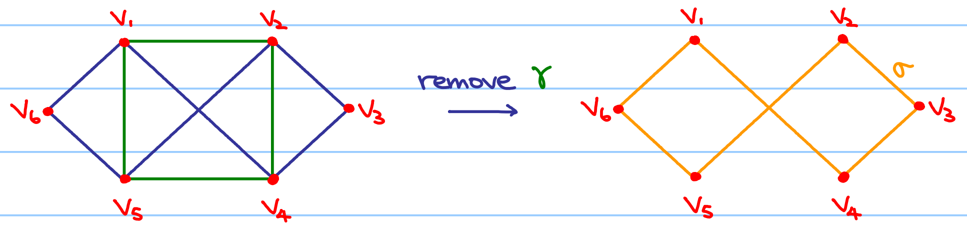
A connected graph G is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Example 10.6



Eulerian Semi-Eulerian Non-Eulerian

Construction of Eulerian trail.



$\gamma : v_1 \to v_2 \to v_4 \to v_5 \to v_1$

$\sigma : v_5 \to v_6 \to v_1 \to v_4 \to v_3 \to v_2 \to v_5$

a vertex in γ

Combining γ and σ : $v_1 \to v_2 \to v_4 \to \underbrace{v_5 \to v_6 \to v_1 \to v_4 \to v_3 \to v_2 \to v_5}_{\sigma} \to v_1$

Definition 10.8

A connected graph G is Hamiltonian if there exists a cycle containing every vertex of G and such cycle is called an Hamiltonian cycle.

A connected, non-Hamiltonian graph G is semi-Hamiltonian if there exists a path containing every vertex of G.

Although it looks like finding Eulerian trails and finding Hamiltonian cycles are similar questions, unfortunately, there is still no characterizations for Hamiltonian graphs. Usually, we can only write down statements with sufficient conditions for a graph to be Hamiltonian.

Proposition 10.3 (Ore, 1960)

If G is a simple graph with $n \geq 3$ vertices, and if $\deg(v) + \deg(w) \geq n$ for each pair of non-adjacent vertices v and w, then G is Hamiltonian.

proof:

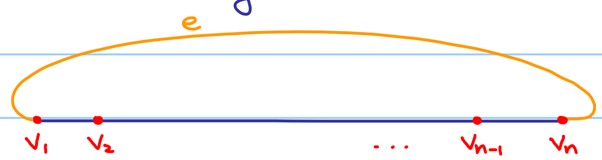
Suppose that G is a simple graph with the property on degree of vertices but it is not Hamiltonian.

We can continue to add more edges until it becomes Hamiltonian.

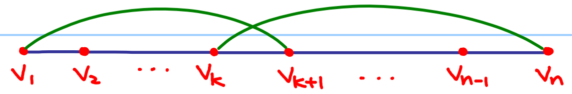
(Think: A complete graph is Hamiltonian.)

Let H be the graph which is the one just before becoming Hamiltonian, i.e. if we add one more edge e to H suitably, it becomes Hamiltonian.

Therefore, with adding e to H, we can construct a Hamiltonian cycle which contains e.



Note that H still has the property that $deg(v)+deg(w) \geq n$ for each pair of non-adjacent vertices v and w of H. Also v_k and v_{k+1} are not adjacent, so $deg(v_1)+deg(v_n) \geq n$ which implies there exists adjacent vertices v_k and v_{k+1} in H such that v_1 is adjacent to v_{k+1} and v_n is adjacent to v_n.



Then, $v_1 \to v_2 \to \cdots \to v_k \to v_n \to v_{n-1} \to \cdots \to v_{k+1} \to v_1$ is a Hamiltonian cycle in H (Contradiction)

Corollary 10.3 (Dirac, 1952)

If G is a simple graph with $n \geq 3$ vertices, and if $deg(v) > \frac{n}{2}$ for all vertices v, then G is Hamiltonian.

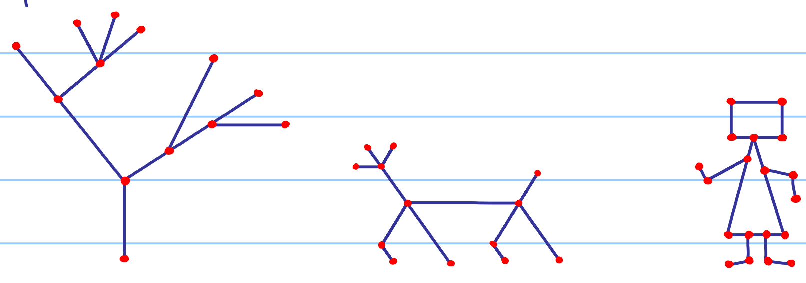
Trees

Definition 10.9

A tree is a connected graph that contains no cycles

A forest is a graph whose components are trees.

Example 10.7



(a) (b) (c)

(a) and (b) are trees while (c) is not.

Proposition 10.4

Let T be a graph with n vertices. Then the following statements are equivalent:

(i) T is a tree ;

(ii) T contains no cycles and has n-1 edges ;

(iii) T is connected and has n-1 edges ;

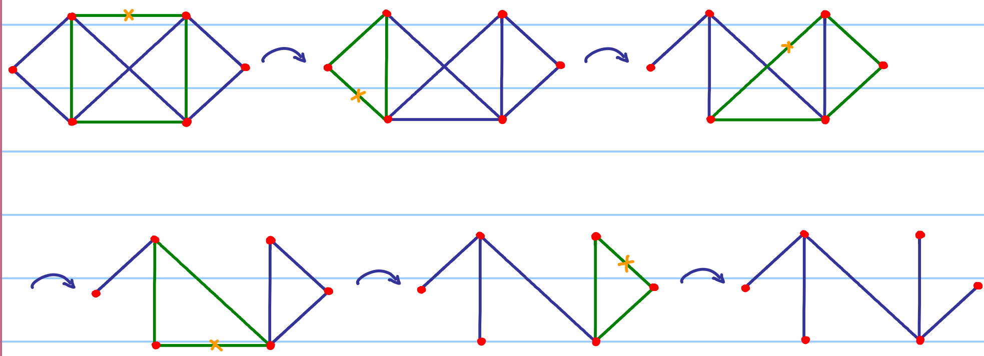
(iv) T is connected and every edge is a bridge, i.e. the graph becomes disconnected if a bridge is removed ;

(v) any two distinct vertices are connected by a unique path,

(vi) T contains no cycles, but the addition of any new edge creates exactly one cycle

Take any connected graph G, suppose that there is a cycle, remove an edge of the cycle and the resulting graph is still connected.
Repeatly doing this until no cycles. Then the remaining graph is a tree that contains all vertices of G, such tree is called a spanning tree of G.

Example 10.7



spanning tree

More Problems on paths and cycles

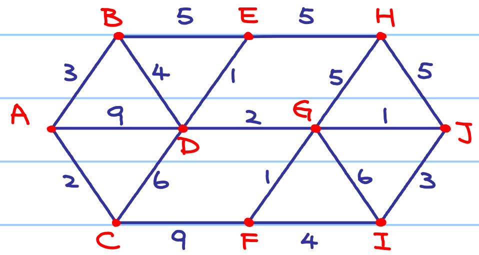
Definition 10.10

A weighted is graph with a number, called weight, assigned to each edge.

The Shortest Path Problem

Given a weighted graph and two distinct vertices, how to find a path with minimum sum of weights from one vertex to the other?

Example 10.8

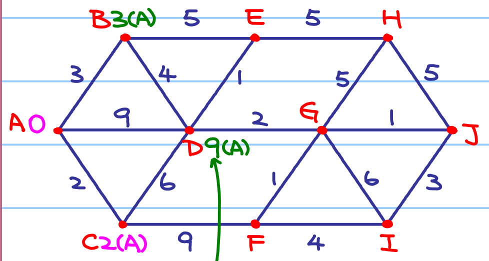


Given a weighted graph which models a network of roads and the weight of an edge is the distance of the corresponding road.

Find the shortest path from A to J.

Dijkstra's Algorithm:

Assign 0 to A and search all vertices adjacent to A:

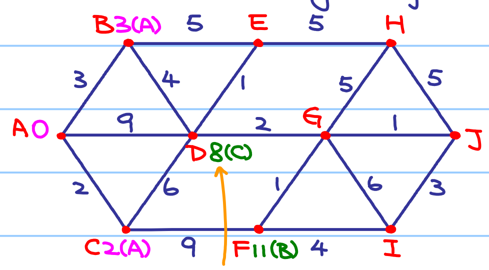


temporary label of shortest path from A to a vertex

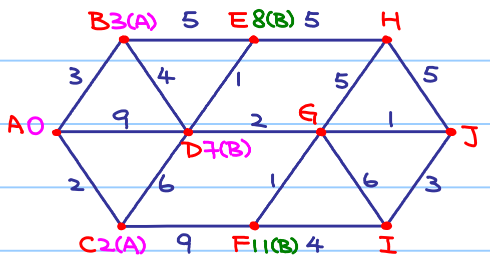
also keep track on the previous vertex

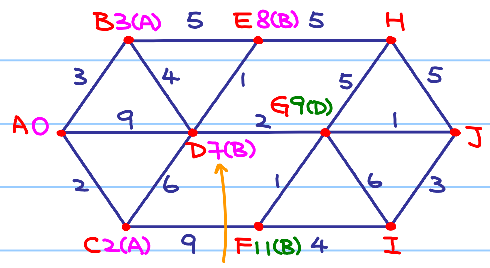
smallest one is confirmed

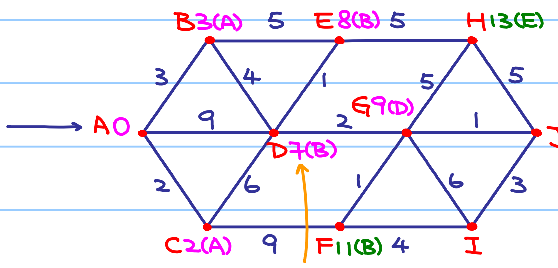
Search all vertices without confirmed label which are adjacent to the vertex with newly confirmed label:

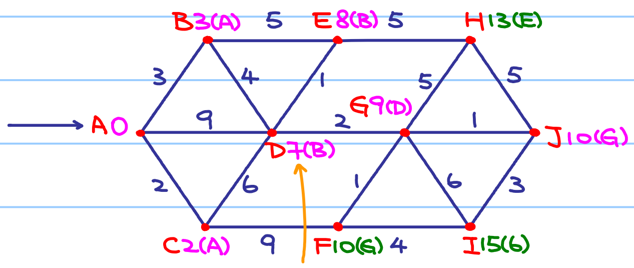


modified since 2+6=8<9









Shortest Path from A to J: A → B → D → G → J

The Chinese Postman Problem

Given a weighted graph, how to travel all the edges at least once and return to the starting point?

If the graph is Eulerian, the Eulerian trail is the solution.

(Think: How about semi-Eulerian graph?)

It can be difficult for a general graph.

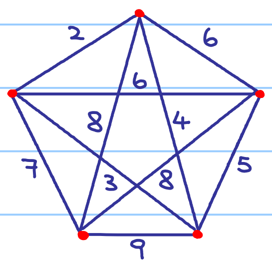
The travelling Salesman Problem

Given a weighted graph, how to travel all the vertices at least once and return to the starting point with minimum sum of weights?

The Minimum Connector Problem

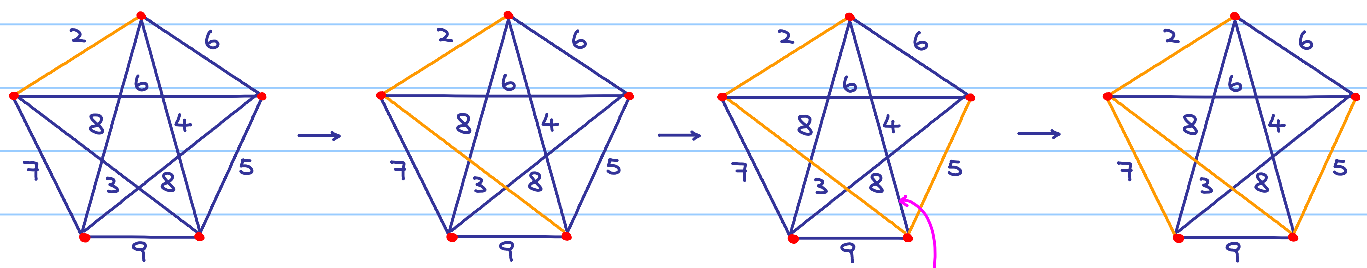
Given a weighted graph, how to find a spanning tree with minimum sum of weights?

Example 10.9



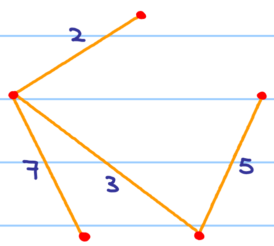
Greedy Algorithm:

Repeatly choosing edges of minimum weight such that no cycle is created:



Do not choose this,

otherwise it forms a cycle.



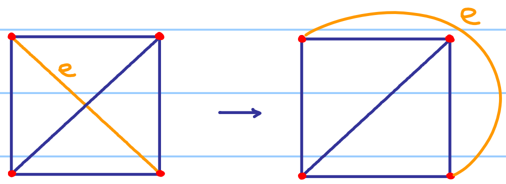
Resulting spanning tree.

Planar Graphs

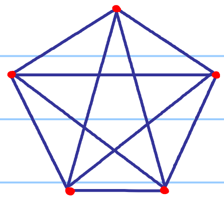
Definition 10.11

A planar graph is a graph that can be drawn in the plane without crossings.

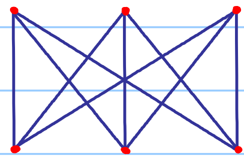
Example 10.10



K_4



K_5



$K_{3,3}$

By moving the edge, we can see K_4 is a planar graph.

How about K_5 and $K_{3,3}$?

Proposition 10.5

K_5 and $K_{3,3}$ are non-planar.

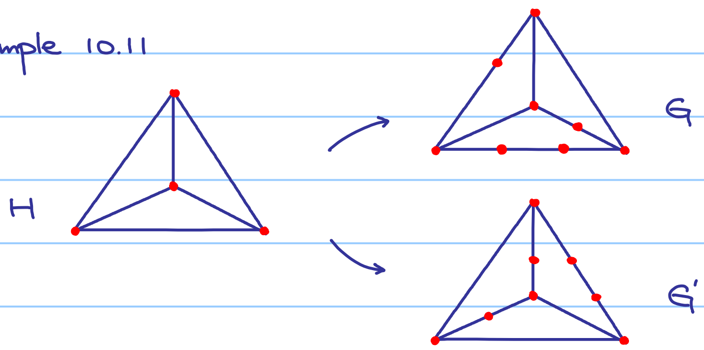
It turns out K_5 and $K_{3,3}$ are "building blocks" of non-planar graphs.

Definition 10.12

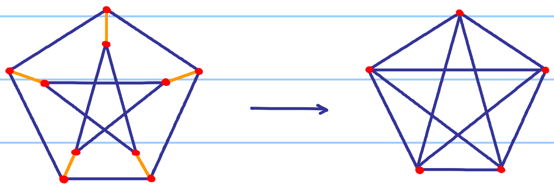
Two graphs are said to be homeomorphic if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges.

A graph G is contractible to another graph G' if we can obtain G' by successively contracting edges of G.

Example 10.11



G and G' are homeomorphic.



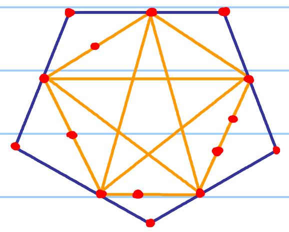
Petersen graph $\qquad$ K_5

By contracting marked edges, we obtain K_5 from the Petersen graph.

Theorem 10.2 (Kuratowski, 1930)

A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

Example 10.12



The graph is non-planar.

Theorem 10.3

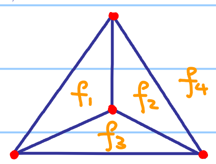
A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.

Example 10.13

The Petersen graph is non-planar as it is contractible to K_5.

Suppose that G is a planar graph, G divides the plane into regions, called faces.

For example:

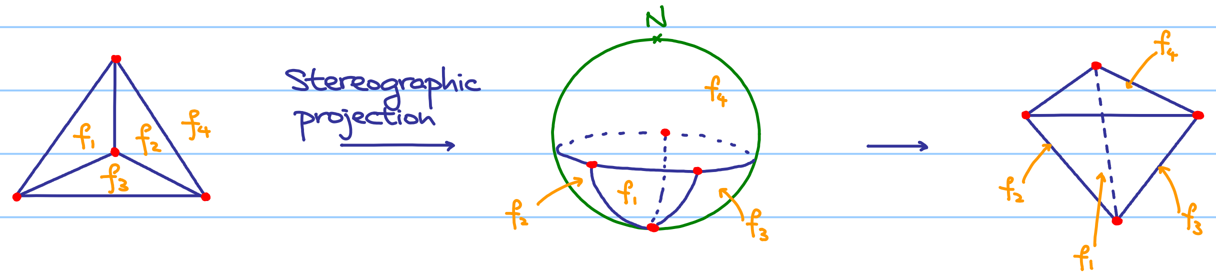


the face f_4 is unbounded, and it is called the infinite face.

Theorem 10.3 (Euler, 1750)

Let G be a plane drawing of a connected planar graph, and let V, E, F be the number of vertices, edges and faces respectively. Then $V-E+F=2$.

What is the relation between a planar graph and a polyhedral surface?



Corollary 10.4

If G is a connected simple planar graph with $V \geq 3$, then $E \leq 3V-6$.

In addition, if G has no triangle, then $E \leq 2V-4$.

proof:

Note that every face is bounded by at least 3 edges, and every edge is shared by two faces, so $3F \leq 2E$. Then result follows from combining it with the Euler's formula.

The proof of the second part is just replacing the above inequality by $4F \leq 2E$.

Exercise 10.1

By using the above corollary, show that

(i) K_5 and $K_{3,3}$ are non-planar

(ii) every simple planar graph contains a vertex of degree at most 5.